Unic2025-2

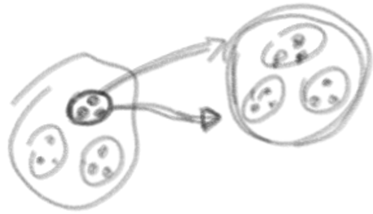
2025年6月11日 22:38

Fractal geometry

Classical analysis : differentiable, continuity, . smooth objects,

fractal dimension, irregular set. Kakeya

Classical fractal sets : self-similar sets



Self-similar sets

Let Λ be a finite index set

Definition A finite family $\Phi = \{f_i\}_{i\in\Lambda}$ of maps $f_i : \mathbb{R}^d \to \mathbb{R}^d$ is called iterated function system (IFS) if each of the maps $f_i, i\in\Lambda$ is a contraction : there exists $0 < r_i < 1$ s.t

$$|f_i(x) - f_i(y)| \le r_i |x-y| \quad x, y \in \mathbb{R}^d$$

attractor of IFS :

Theorem For any IFS $\Phi = \{f_i\}_{i\in\Lambda}$, there exists a compact $X \neq \phi$ s.t

$$X = \bigcup_{i\in\Lambda} f_i(X)$$

Proof : Let $\mathcal{X} = \{X \subset \mathbb{R}^d : X \neq \phi \text{ is compact}\}$. Then $\mathcal{X}$ becomes a complete metric space with the Hausdorff metric

$$d_H(X, Y) := \inf\{\varepsilon > 0 : X \subset Y^{(\varepsilon)}, Y \subset X^{(\varepsilon)}\}$$

where $Y^{(\varepsilon)} = \{x \in \mathbb{R}^d : \mathrm{dist}(x, Y) < \varepsilon\}$ is the ε-neighborhood of Y.

where $Y^{\varepsilon} = \{x \in \mathbb{R}^n : \mathrm{dist}(x, Y) < \varepsilon\}$ is the ε-neighborhood of Y.

Define $\Phi : \mathcal{X} \longrightarrow \mathcal{X}$.

$$\Phi(X) := \bigcup_{i \in \Lambda} f_i(X), \quad X \in \mathcal{X}$$

Then Φ satisfies.

Exercise | $d_H(\Phi(X), \Phi(Y)) \le \underbrace{\left(\max_{i\in\Lambda} r_i\right)}_{0 < \quad < 1} d_H(X, Y), \quad X, Y \in \mathcal{X}$

Since $\max_{i\in\Lambda} r_i < 1$, the map Φ is a contraction on the complete metric space $(\mathcal{X}, d_H)$

Hence by the Banach fixed pt theorem, $\exists\,!\ X \in \mathcal{X}$ satisfying

$$\Phi(X) = X$$

$$\iff \quad X = \bigcup_{i\in\Lambda} f_i(X)$$

the set X is the attractor of Φ

Definition. A contraction $f_i : \mathbb{R}^d \to \mathbb{R}^d$ is called a similitude if

$$|f_i(x) - f_i(y)| = r_i |x - y| \quad x, y \in \mathbb{R}^d$$

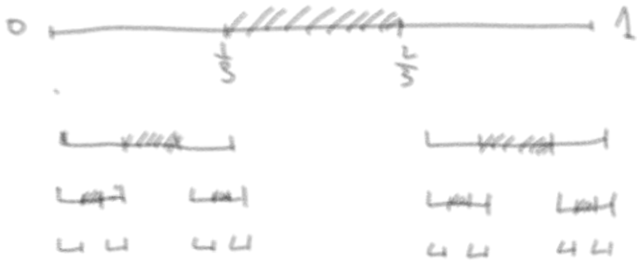
which is equivalent to say that $f_i(x) = r_i A_i x + a_i$ for some $A_i \in O(d)$ and $a_i \in \mathbb{R}^d$.

An IFS $\Phi = \{f_i\}_{i\in\Lambda}$ is selfsimilar if each $f_i \in \Phi$ is a similitude

The attractor X for a selfsimilar IFS is called a selfsimilar set

$$\underline{X} = \bigcup_{i \in I} f_i(X)$$

Examples: (1) The classical $\frac{1}{3}$-Cantor set



$\frac{1}{3}$-Cantor set = attractor of the IFS $\Phi = \{f_1, f_2\}$

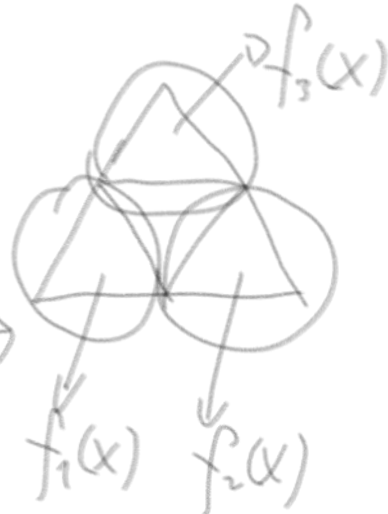
$$f_1(x) = \frac{1}{3}x \quad , \quad f_2(x) = \frac{1}{3}x + \frac{2}{3}$$

$C_{\frac{1}{3}}$

$$C_{\frac{1}{3}} = f_1(C_{\frac{1}{3}}) \cup f_2(C_{\frac{1}{3}})$$
$$= \left(\frac{1}{3} \times C_{\frac{1}{3}}\right) \cup \left(\frac{1}{3} \times C_{\frac{1}{3}} + \frac{2}{3}\right)$$

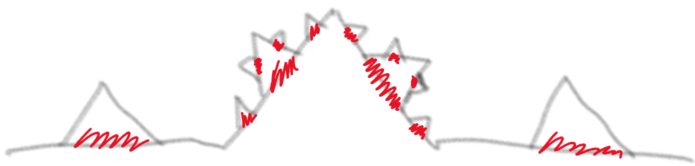
(2) The Sierpinski gasket

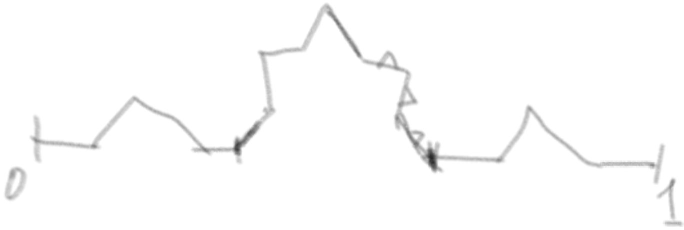
It is the attractor of the IFS $\{f_1, f_2, f_3\}$



$$f_1(x) = \frac{1}{2}x$$
$$f_2(x) = \frac{1}{2}x + \left(\frac{1}{2}, 0\right), \quad f_3(x) = \frac{1}{2}x + \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$$

(3) The Koch snowflake curve





~~attractor~~ of $\{f_1, f_2, f_3, f_4\}$

$$f_1(x) = \tfrac{1}{3}x, \quad f_2(x) = \tfrac{1}{3}A^2x + (\tfrac{1}{2}, 0)$$

$$f_3(x) = \tfrac{1}{3}Ax + (\tfrac{1}{3}, 0) \quad f_4(x) = \tfrac{1}{3}x + (\tfrac{2}{3}, 0))$$

$A \in O(2)$ is the counter-clockwise ~~rot~~ rotation by $60°$

Cylinder sets and iterators

Assume $d = 1$, $f_i(x) = r_i x + a_i$, $r_i = r$ $\forall i \in \Lambda$ (all maps contract with the same contraction ~~ratio~~ rate)

$$f_i(x) = rx + a_i \quad 0 < r < 1, \; a_i \in \mathbb{R}$$

Λ = alphabet set $\quad \Lambda^n = \{ \underline{i_1 i_2 \cdots i_n} : i_1, i_2, \ldots, i_n \in \Lambda \}$, $n \in \mathbb{N}$

"words of length n"

For a word $\underline{i} = \underline{i_1 i_2 \cdots i_n} \in \Lambda^n$, denote the composition

$$f_{\underline{i}} = \underline{f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}}$$

$$f_j(x) = rx + a_j, \; j \in \Lambda$$

In our case: $f_{\underline{i}}(x) = r^n x + \sum^{n} a_{i_k} r^{k-1}$ ...

In our case: $f_i(x) = r^n x + \sum_{k=1}^{n} a_{i_k} r^{k-1} = r^n x + a_i$

where $a_i = \sum_{k=1}^{n} a_{i_k} r^{k-1} = f_i(0)$

The set $f_i(X)$ is called an nth generation cylinder-set associated to the word $i \in \Lambda^n$

$$X = \bigcup_{i\in\Lambda} f_i(X) = \bigcup_{i\in\Lambda} f_i\Big(\bigcup_{j\in\Lambda} f_j(X)\Big)$$

$$= \bigcup_{i\in\Lambda}\bigcup_{j\in\Lambda} f_i(f_j(X))$$

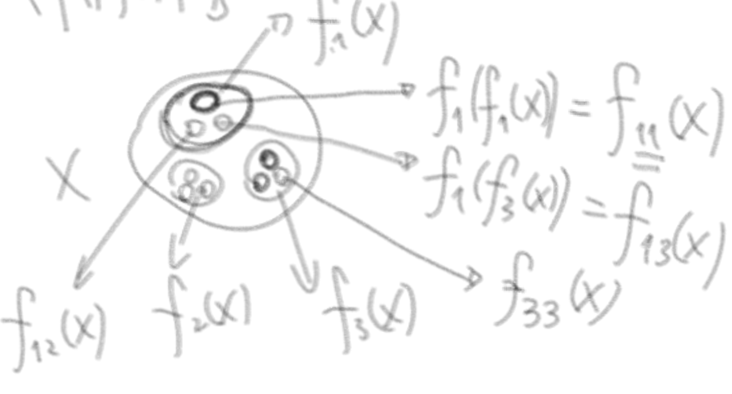
$$= \bigcup_{i\in\Lambda^2} f_i(X) \qquad i\in\Lambda \quad f_i \qquad i\in\Lambda^2 \quad f_i$$

$$= \bigcup_{i\in\Lambda^2} f_i\Big(\bigcup_{j\in\Lambda} f_j(X)\Big)$$

$$= \bigcup_{i\in\Lambda^3} f_i(X)$$

$$= \vdots = \bigcup_{i\in\Lambda^n} f_i(X) \quad n\in\mathbb{N}$$

$\{f_1, f_2, f_3\}$



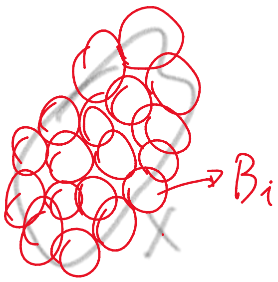
Fractal dimension

Different dimensions are standard way to measure how "big" or how much "space" a set X occupies

Def The covering number of a bounded set X at the scale $\varepsilon>0$ is defined by

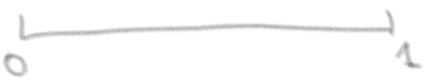
$$N_\varepsilon(X) := \min\{k\in\mathbb{N} : \text{we can cover } X \text{ by } k \text{ closed balls } B_i \text{ of } \operatorname{diam}(B_i)=\varepsilon\}$$

$$= \min\{k\in\mathbb{N} : \exists B_1,\dots,B_k \text{ s.t. } X\subset \bigcup_{i=1}^{k} B_i,\ \operatorname{diam}(B_i)=\varepsilon\}$$



$\varepsilon = \frac{1}{n}$

Topological dimension -



$\dim_T = 1$

$\dim_T = 2$

X_3 $\dim_T = 3$

$N_\varepsilon(X_1) \approx \left(\frac{1}{\varepsilon}\right)$

$N_\varepsilon(X_2) \approx \left(\frac{1}{\varepsilon}\right)^2$

$N_\varepsilon(X_3) \approx \left(\frac{1}{\varepsilon}\right)^3$

ε $0<\alpha<1$

$N_\varepsilon(C_{\frac{1}{3}}) \approx \left(\frac{1}{\varepsilon}\right)^{\alpha}$ $\frac{\log 2}{\log 3}$

Def The Box dimension (Minkowski dim) of X is the exponential growth rate of the covering numbers $N_\varepsilon(X)$

$$\dim_B X = \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(X)}{\log(\frac{1}{\varepsilon})} = s$$

$$N_\varepsilon(X) \approx \left(\frac{1}{\varepsilon}\right)^{s+o(1)}$$

provided that the limit exists

upper Box dim: $\overline{\dim}_B X = \limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon(X)}{\log \frac{1}{\varepsilon}}$

lower Box dim $\underline{\dim}_B(X) = \liminf$:

Example: (1) $X = [0,1]$, $N_\varepsilon(X) \approx \left(\frac{1}{\varepsilon}\right)^1 \Rightarrow \dim_B X = 1$

(2) For $X = C_{\frac{1}{3}}$ $\quad \dim_B X = \frac{\log 2}{\log 3}$

(using the cylinder sets for the optimal covering)

$\left(\frac{1}{3}\right)^2 \to 4$

$\left(\frac{1}{3}\right)^3 \to 8$

$\vdots$

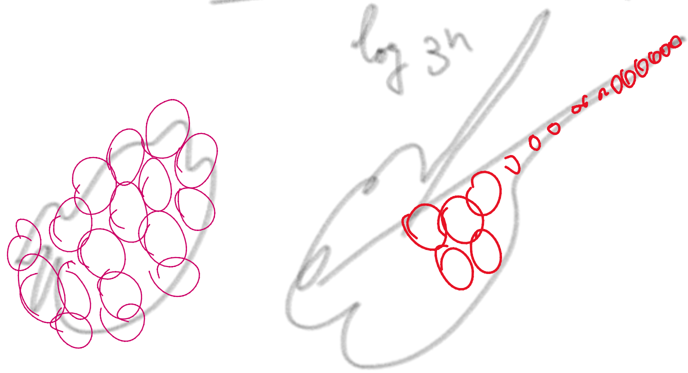
$\left(\frac{1}{3}\right)^n \to 2^{n-1}$

$$N_{(\frac{1}{3})^n}\left(C_{\frac{1}{3}}\right) \simeq 2^{n-1}$$

$\varepsilon = \left(\frac{1}{3}\right)^n \qquad \log N_{(\frac{1}{3})^n}\left(C_{\frac{1}{3}}\right)$

$\varepsilon = \left(\frac{1}{3}\right)^n$

$$\frac{\log N_{\left(\frac{1}{3}\right)^n}(C_{\frac{1}{3}})}{\log 3^n} \longrightarrow \frac{\log 2}{\log 3}$$



The value of Box dim is evaluated from coverings that have fixed upper bounds for the size (diameter). The optimal coverings for a set may not be given by such covers

The notion of Hausdorff dimension $\dim_H X$ takes this into account

$$\dim_H X = \inf \left\{ s>0 : \forall \varepsilon>0 \text{ we can cover } X \text{ by balls } B_i \text{ satisfying } \sum_i \operatorname{diam}(B_i)^s < \varepsilon \right\}$$

Exercise: $\dim_H X \leq \underline{\dim}_B X$

Theorem (Falconer) If X is selfsimilar, then $\dim_H X = \dim_B X$

Bounding dimension of selfsimilar sets

X = selfsimilar set associated to $\Phi = \{f_i(x) = rx + a_i\}_{i \in \Lambda}$

Recall $X = \bigcup_{i \in \Lambda} f_i(X)$

$i \in \Lambda$

$$= \bigcup_{i \in \Lambda^n} f_i(X) \qquad X = \bigcup_{i \in \Lambda^n} f_i(X)$$

If $i \in \Lambda^n$, then $f_i(x) = r^n x + a_i$ for some $a_i \in \mathbb{R}$

$$f_i(X) = r^n(X) + a_i$$

$$\Rightarrow \operatorname{diam}(f_i(X)) \leqq r^n \times \operatorname{diam}(X)$$

$$\varepsilon = r^n \times \operatorname{diam}(X)$$

$\Rightarrow N_\varepsilon(X) \leq$ Cardinality of $\Lambda^n = |\Lambda|^n$

$$\dim_B(X) \leq \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(X)}{\log \frac{1}{\varepsilon}} \leq \lim_{n \to \infty} \frac{\log |\Lambda|^n}{\log\left(r^n \times \underbrace{\operatorname{diam}(X)}_{c_0}\right)^{-1}}$$

$$f_i(X) \cap f_j(X) = \emptyset \qquad = \frac{\log |\Lambda|}{\log \frac{1}{r}} = \dim_S X$$

similarity dimension of X