Fourier Analysis - 18/6/2025

Denote the functions of moderate decrease by $\mathcal{M}(\mathbb{R})$.

Definition 2 If $f \in \mathcal{M}(\mathbb{R})$, then the Fourier transform of f is the function $\hat{f}$ defined by

$$\hat{f}(x) = \int_{\mathbb{R}} f(t) e^{-2\pi i x t}\, dt$$

for all $x \in \mathbb{R}$.

~~Theorem~~ 3

Notation: For $f \in \mathcal{M}(\mathbb{R})$, denote

$$\|f\|_1 = \int_{\mathbb{R}} |f(t)|\, dt$$

For any bounded function f,

$$\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)|$$

Theorem 3 For $f \in \mathcal{M}(\mathbb{R})$,

(a) $\|\hat{f}\|_\infty \leq \|f\|_1$

(b) $\hat{f}$ is uniformly continuous.

Proof

(a) For any $x \in \mathbb{R}$

$$|\hat{f}(x)| = \left| \int_{\mathbb{R}} f(t) e^{-2\pi i x t}\, dt \right| = \left| \lim_{N\to\infty} \int_{-N}^{N} f(t) e^{-2\pi i x t}\, dt \right|$$

$$= \lim_{N\to\infty} \left| \int_{-N}^{N} f(t) e^{-2\pi i x t}\, dt \right| \leq \lim_{N\to\infty} \int_{-N}^{N} |f(t)|\, |e^{-2\pi i x t}|\, dt$$

$= \|f\|_1$

So for all $x \in \mathbb{R}$, we have $|\hat{f}(x)| \leq \|f\|_1$

$$\Rightarrow \|\hat{f}\|_\infty = \sup_{x \in \mathbb{R}} |\hat{f}(x)| \leq \|f\|_1 .$$

(b) Given $\varepsilon > 0$, for any $x, y \in \mathbb{R}$

$$|\hat{f}(x) - \hat{f}(y)| = \left| \int_{\mathbb{R}} f(t) \left(e^{-2\pi i x t} - e^{-2\pi i y t} \right) dt \right|$$

$$\leq \left| \int_{|t|>M} f(t) \left(e^{-2\pi i x t} - e^{-2\pi i y t} \right) dt \right| + \left| \int_{|t|<M} f(t) \left(e^{-2\pi i x t} - e^{-2\pi i y t} \right) dt \right|$$

$$\leq \int_{|t|>M} |f(t)| \left(|e^{-2\pi i x t}| + |e^{-2\pi i y t}| \right) dt$$

$$+ \int_{|t|<M} |f(t)| \, | e^{-2\pi i x t} - e^{-2\pi i y t} | \, dt$$

$$\leq 2 \int_{|t|>M} |f(t)| \, dt + 2\pi \int_{|t|<M} |f(t)| \, |t| \, |x-y| \, dt$$

$$\leq 2 \int_{|t|>M} |f(t)| \, dt + 2\pi M |x-y| \, \|f\|_1$$

Choose $M = M(\varepsilon)$ such that $\int_{|t|>M} |f(t)| \, dt < \frac{\varepsilon}{4}$
(by Proposition 1)

and choose $\delta = \dfrac{\varepsilon}{4\pi M \|f\|_1}$

Then for $|x-y| < \delta$, we have $|\hat{f}(x) - \hat{f}(y)| < \varepsilon$ $\square$

<u>Example</u> Compute the Fourier transform of $f = \chi_{[a,b]}$

i.e. $f(t) = \begin{cases} 1 & \text{if } t \in [a,b] \\ 0 & \text{otherwise} \end{cases}$.

Then $\hat{f}(x) = \int_a^b e^{-2\pi i x t}\, dt =$

$$= -\frac{1}{2\pi i x}\left(e^{-2\pi i x b} - e^{-2\pi i x a} \right)$$

In particular, $|\hat{f}(x)| \leq \frac{1}{\pi |x|}$

So $|\hat{f}(x)| \longrightarrow 0$ as $|x| \to \infty$.

Other properties of the Fourier transform

<u>Theorem 4</u> Let $f \in \mathcal{M}(\mathbb{R})$

(i) Let $g(t) = f(t+h)$ (translation), $h \in \mathbb{R}$.

Then $\hat{g}(x) = e^{2\pi i h x} \hat{f}(x)$.

(ii) Let $g(t) = e^{-2\pi i t h} f(t)$ (modulation), $h \in \mathbb{R}$

Then $\hat{g}(x) = \hat{f}(x+h)$.

(iii) Let $g(t) = f(at)$ (dilation), $a > 0$

Then $\hat{g}(x) = a^{-1} \hat{f}(a^{-1} x)$.

Proof

(i) $\hat{g}(x) = \int_{\mathbb{R}} g(t)\, e^{-2\pi i x t}\, dt = \int_{\mathbb{R}} f(t+h) e^{-2\pi i x t} dt$

$= e^{2\pi i x h} \int_{\mathbb{R}} f(t) e^{-2\pi i x t} dt = e^{2\pi i x h}\, \hat{f}(x)$.

(ii) $\hat{g}(x) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i t h} e^{-2\pi i t x} dt$

$= \int_{\mathbb{R}} f(t)\, e^{-2\pi i t (x+h)} dt = \hat{f}(x+h)$

(iii) $\hat{g}(x) = \int_{\mathbb{R}} f(at)\, e^{-2\pi i x t} dt = \frac{1}{a} \int_{\mathbb{R}} f(t)\, e^{-2\pi i \frac{xt}{a}} dt$

$= \frac{1}{a} \hat{f}\left(\frac{x}{a}\right)$.

Theorem 5 Suppose $f \in \mathcal{M}(\mathbb{R})$ and $tf(t) \in \mathcal{M}(\mathbb{R})$. Then $\hat{f}$ is differentiable and $\frac{d\hat{f}}{dx}(x) = -2\pi i\, \widehat{(tf(t))}(x)$.

Proof To show that $\hat{f}$ is differentiable, let $\varepsilon > 0$ and consider

$$\frac{\hat{f}(x+h) - \hat{f}(x)}{h} - \left(-2\pi i\, \widehat{(tf(t))}(x)\right)$$

$$= \int_{\mathbb{R}} f(t)\, e^{-2\pi i t x} \left(\frac{e^{-2\pi i t h} - 1}{h} + 2\pi i t \right) dt = \int_{|t|<N} + \int_{|t|>N}$$

$\Rightarrow |f(t)| \le M$ for some M.

Since $f(t)$ and $tf(t)$ are ~~in~~ in $\mathcal{M}(\mathbb{R})$ we can find $N>0$ such that

$$\int_{|t|>N} |f(t)|\,dt < \varepsilon \text{ and } \int_{|t|>N} |t|\,|f(t)|\,dt < \varepsilon$$

There exist h_0 such that for $|h|<|h_0|$

$$\left| \frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right| \le \frac{\varepsilon}{MN} \text{ for } |t| \le N.$$

So $\left| \frac{\hat{f}(x+h) - \hat{f}(x)}{h} - \left(-2\pi i \widehat{tf(t)}(x)\right) \right|$

$$\le \left| \int_{|t|<N} \right| + \left| \int_{|t|>N} \right|$$

$$\le \int_{|t|<N} \left| f(t)\, e^{-2\pi i t x} \left(\frac{e^{-2\pi i t x} - 1}{h} + 2\pi i x \right) \right| dt$$

$$+ C\varepsilon \le C'\varepsilon. \qquad \square$$

<u>Theorem 6</u> Let f ~~& bounded~~ $\in \mathcal{M}(\mathbb{R})$ and f be differentiable. ~~Then~~ and ~~($f' \in \mathcal{M}(\mathbb{R})$)~~

Then $\widehat{f'}(x) = 2\pi i x \hat{f}(x)$

<u>Proof</u> $\int_{-N}^{N} f'(t)\, e^{-2\pi i t x}\, dt$ by Integration by Parts,

$$= \underbrace{\left[f(t)\, e^{-2\pi i t x} \right]_{-N}^{N}}_{N\to\infty,\ 0} + 2\pi i x \int_{-N}^{N} f(t)\, e^{-2\pi i t x}\, dt$$

$$\Rightarrow \lim_{N\to\infty} \int_{-N}^{N} f'(t)\, e^{-2\pi i t x}\, dt = 2\pi i x \lim_{N\to\infty} \int_{-N}^{N} f(t)\, e^{-2\pi i t x}\, dt$$