Fourier Analysis 11/6/2025

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Fourier Analysis - An introduction.

To recap Riemann Integration
see Appendix: Section 1
Section 2.3

~~Define~~ Consider continuous functions f on $\mathbb{R}$. $N \in \mathbb{R}$.

Let $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N \to \infty} \int_{-N}^{N} f(x)\,dx$

The limit may not exist.
Take e.g. $f(x) = 1$ or $f(x) = \frac{1}{1+|x|}$.

A function f defined on $\mathbb{R}$ is said to be of <u>moderate decrease</u> if f is continuous and there exists a constant $A > 0$ so that

$$|f(x)| \leq \frac{A}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

For example any $f(x) = \frac{1}{1+|x|^n}$ for $n \geq 2$ is of moderate decrease.

$f(x) = e^{-a|x|}$ is also of moderat decrease

Let $I_N = \int_{-N}^{N} f(x)\,dx$

Want to show that $\{I_N\}$ is a Cauchy sequence.

$M > N$.

$$|I_M - I_N| = \left| \int_{-M}^{M} f(x)\,dx - \int_{-N}^{N} f(x)\,dx \right|$$

$$= \left| \int_{-M}^{-N} f(x)\,dx + \int_{N}^{M} f(x)\,dx \right|$$

$$\leq \int_{N \leq |x| \leq M} |f(x)|\,dx$$

$$\leq A \int_{N \leq |x| \leq M} \frac{1}{1+x^2}\,dx \leq A \int_{N \leq |x| \leq M} \frac{1}{x^2}\,dx$$

$$\leq 2A \left(\frac{1}{N} - \frac{1}{M} \right)$$

$$\leq \frac{2A}{N} \longrightarrow 0 \text{ as } N \to \infty.$$

Hence $\{I_N\}$ is Cauchy and therefore the limit $\lim_{N \to \infty} I_N$ exists.

<u>Proposition 1</u> The integral of a function of moderate decrease satisfies the following properties:

(i) Linearity: if f, g are of moderate decrease and $a, b \in \mathbb{C}$, then

$$\int_{-\infty}^{\infty} af(x) + bg(x)\,dx = a \int_{-\infty}^{\infty} f(x)\,dx + b \int_{-\infty}^{\infty} g(x)\,dx.$$

(ii) Translation invariance: for every $h \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} f(x-h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

(iii) Scaling under dilations: if $\delta > 0$, then $\delta \int_{-\infty}^{\infty} f(\delta x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$

(iv) Continuity: If f is of moderate decrease, then

$$\int_{-\infty}^{\infty} |f(x-h) - f(x)|\,dx \longrightarrow 0 \text{ as } h \to 0.$$

Proof. (i) Just linearity of the limit and the Riemann integral over a bounded interval.

(ii) Suffices to show that

$$\lim_{N\to\infty} \left| \int_{-N}^{N} f(x-h)\,dx - \int_{-N}^{N} f(x)\,dx \right| = 0$$

Note $\int_{-N}^{N} f(x-h)\,dx = \int_{-N-h}^{N-h} f(x)\,dx$

$$= \left| \int_{-N-h}^{-N} f(x)\,dx + \int_{N-h}^{N} f(x)\,dx \right| \quad \text{(for } h>0\text{)}$$

sim. for $h<0$.

$$\leq \frac{A'}{N^2} \longrightarrow 0 \text{ as } N \to \infty.$$

(iii) Similar. Observe that

$$\delta \int_{-N}^{N} f(\delta x)\,dx = \int_{-\delta N}^{\delta N} f(x)\,dx \quad \text{(Change of variables)}$$

(iv) Take $|h| \leq 1$. For given $\varepsilon > 0$, choose N large enough such that

$$\int_{|x| \geq N} |f(x)|\,dx \leq \frac{\varepsilon}{4}.$$

Then $$\int_{|x| \geq N+1} |f(x)|\,dx \leq \int_{|x| \geq N} |f(x)|\,dx \leq \frac{\varepsilon}{4}$$

and $$\int_{|x| \geq N+1} |f(x-h)|\,dx \leq \int_{|x| \geq N} |f(x)|\,dx \leq \frac{\varepsilon}{4}$$

for any $|h| \leq 1$.

Fixing this N we have that since f is continuous, it is uniformly continuous on the interval $[-N-1, N+1]$.

Hence $\sup_{|x| \leq N+1} |f(x-h) - f(x)| \leq \frac{\varepsilon}{4(N+1)}$ for all h small enough.

Putting everything together:

$$\int_{-\infty}^{\infty} |f(x-h) - f(x)|\,dx \leq \int_{-N-1}^{N+1} |f(x-h) - f(x)|\,dx$$

$$+ \int_{|x| \geq N+1} |f(x-h)|\,dx + \int_{|x| \geq N+1} |f(x)|\,dx$$

$$\leq \frac{\varepsilon}{4(N+1)} 2(N+1) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon \qquad \square$$