UNIC Mathematics Summer School, June 2025

In this summer school, a number of areas of mathematics are introduced as follows.

Topic I: Theory of Stochastic Processes

Instructor: Prof. Dr. Andrzej Łuczak, Łodz University

1. **Course description**: The aim of the course is to present some elements of the theory of stochastic processes such as continuity with probability one, continuity in probability and continuity in the mean of a process, strong and weak equivalence of processes, finite-dimensional distributions of a process, Kolmogorov existence theorem and its implications, processes with independent increments, Wiener and Poisson processes.

2. **Course goals:**
   - To provide the students with basic knowledge of the theory of stochastic processes allowing them to study more advanced topics such as e.g. stochastic integration.

3. **Timetable**

Date & Time	Content
Virtual phase	
11.06.2025 or 12.06.2025 (2 hrs) exact time to be announced later (TBA)	**Lecture 1: Review of Course Material** Presentation of the subject of the course and its connection with other fields of mathematics. Overview of the students' mathematical background necessary for the understanding of the lectured material.
18.06.2025 (2 hrs) exact time TBA	**Lecture 2: Introduction to the Topic** Brief recalling of basic notions of probability and some more advanced topics such as e.g. convergence with probability one and in probability, and their mutual relations.
Physical phase at Koc University	
23.06.2025 (2 hrs) 9:00-11:00	**Lecture I:** Definitions of a stochastic process and their mutual relations. Stochastic process as an "infinite dimensional random variable".
24.06.2025 (2 hrs) 11:00-13:00	**Lecture II:** Continuity of a stochastic process. Finite-dimensional distributions of a process and equivalence of processes. Processes of the second order. Correlation function of a second order process.
25.06.2025 (2 hrs) 9:00-11:00	**Lecture III:** Kolmogorov theorem on the existence of a stochastic process with given finite dimensional distributions and its consequences.
26.06.2025 (2 hrs) 15:00-17:00	**Lecture IV:** Processes with independent increments. Poisson and Wiener processes.

Recommended reading/textbook list for the course:

- P. Billingsley "Probability and Measure", Wiley, New York, 1979.
- M. Loève “Probability Theory II, Springer, New York, 1978.

4. **Mode of delivery:**

- Students need to read the materials and complete the problem assignments.
- Virtual Seminars via Zoom or Microsoft Teams.
- Face-to-face classes at Koc University.
- Virtual and face to face group meetings

5. **Period**: June 2025

   Virtual phase: 11.06-12.06 and 18.06.2025
   Physical phase: 22.06 - 29.06.2025

6. **Students:** Open to undergraduate students who have received at least 60 ECTS credits in mathematical courses or for graduate students.

7. **Workload**:

   - Independent work (Reading materials, homework problems) 20 h
   - Lectures 12 h (Virtual 4 h, Face to face 8 h)
   - Total number of hours: 32 h

8. **Assessment methods:**

   In order to pass this course, the following assessment criteria will be used:
   - Compulsory participation in the virtual and physical lectures
   - Compulsory reading and homework assignments

Grading: Pass/Fail or Satisfactory/Unsatisfactory

9. **Teaching language:** English

10. **ECTS:** 5 in total together with the other three topics:
    - Additive combinatorics
    - Diophantine m-tuples
    - Fourier analysis

in the Summer School. Total workload of the four topics is 4x32=128 h.

Topic II: Fourier Analysis

Instructor: Dr. Spyridon Dendrinos, University College Cork

1. **Course description**: The course aims to explore fundamental properties of the Fourier transform and convolution which are key operations in Harmonic Analysis.

2. **Course goals:**
   - To develop the connection between the smoothness of a function and the decay of its Fourier transform at infinity.
   - To show why this leads to definition of the Schwarz space and study the behaviour of the Fourier transform on the Schwarz space.
   - To define and obtain basic properties of convolution and its interplay with the smoothness of the functions involved and with the Fourier transform.
   - To explore basic properties of Fourier inversion.

3. **Timetable**

Date & Time	Content
Virtual phase	
11.06.2025 or 12.06.2025 (2 hrs) exact time to be announced later (TBA)	**Lecture 1: Review of Course Material** Presentation of the subject of the course and its connection with other fields of mathematics. Overview of the students' mathematical background necessary for the understanding of the lectured material.
18.06.2025 (2 hrs) exact time TBA	**Lecture 2: Introduction to the Topic** Definition of the Fourier transform for functions on the real line. Basic properties. The Fourier transform of derivatives of a function.
Physical phase at Koc University	
23.06.2025 (2 hrs) 11:00-13:00	**Lecture I:** Interplay between smoothness of a function and the decay of its Fourier transform at infinity. The Schwartz space and its properties. The Fourier transform of Schwartz functions.
24.06.2025 (2 hrs) 9:00-11:00	**Lecture II:** Definition of convolution and its properties. The Fourier transform of a convolution of two functions. Families of good kernels.
25.06.2025 (2 hrs) 11:00-13:00	**Lecture III:** Fourier inversion. Plancherel Theorem. Applications to Partial Differential Equations.
27.06.2025 (2 hrs) 15:00-17:00	**Lecture IV:** The uncertainty principle in Fourier Analysis – various viewpoints.

Recommended reading/textbook list for the course:

- E.M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.

- E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

4. Mode of delivery:

- Students need to read the materials and complete the problem assignments.
- Virtual Seminars via Zoom or Microsoft Teams.
- Face-to-face classes at Koc University.
- Virtual and face to face group meetings

5. Period: June 2025

Virtual phase: 11.06-12.06 and 18.06.2025
Physical phase: 22.06 - 29.06.2025

6. Students: Open to undergraduate students who have received at least 60 ECTS credits in mathematical courses or for graduate students.

7. Workload:

- Independent work (Reading materials, homework problems) 20 h
- Lectures 12 h (Virtual 4 h, Face to face 8 h)
- Total number of hours: 32 h

8. Assessment methods:

In order to pass this course, the following assessment criteria will be used:

- Compulsory participation in the virtual and physical lectures
- Compulsory reading and homework assignments

Grading: Pass/Fail or Satisfactory/Unsatisfactory

9. Teaching language: English

10. ECTS: 5 in total together with the other three topics:

- Additive Combinatorics
- Theory of Stochastic Processes
- Diophantine m-tuples

in the Summer School. Total workload of the four topics is 4x32=128 h.

Topic III: Additive Combinatorics

Instructor: Prof. Dr. Meng Wu, Oulu University

1. **Course description**: Additive combinatorics is the study of additive structures within sets. A fundamental topic is understanding what can be inferred about the structures of two sets A and B, given that the size of their sumset A+B is small. The field has deep connections to various areas of mathematics. In this course, we will see one such connection, on fractal geometry and the fractal dimension of self-similar fractals.

2. **Course goals:**
   - Introduce students to key topics in additive combinatorics.
   - Explore applications of additive combinatorial methods in fractal geometry, particularly in the dimension theory of self-similar sets.

3. **Timetable**

Date & Time	Content
	Virtual phase
11.06.2025 or 12.06.2025 (2 hrs) exact time to be announced later (TBA)	**Lecture 1: Review of Course Material** Presentation of the subject of the course and its connection with other fields of mathematics. Overview of the students' mathematical background necessary for the understanding of the lectured material.
18.06.2025 (2 hrs) exact time TBA	**Lecture 2: Introduction to the Topic** Examples of additive combinatorics, basic objects in fractal geometry, examples of self-similar fractals
	Physical phase at Koc University
23.06.2025 (2 hrs) 15:00-17:00	**Lecture I:** Basic definitions, examples, Cauchy-Davenport theorem, Freiman-Ruzsa theorem.
26.06.2025 (2 hrs) 9:00-11:00	**Lecture II:** Sum-product phenomena, inverse theroems of Bourgain, Hochman.
27.06.2025 (2 hrs) 11:00-13:00	**Lecture III:** Basic notion in fractal geometry, fractal dimensions, dimensions of self-similar sets
28.06.2025 (2 hrs) 9:00-11:00	**Lecture IV:** Application of additive combinatorics to dimension theory of self-similar sets

Recommended reading/textbook list for the course:

- T. Tao and V.H. Vu, *Additive Combinatorics*, Cambridge studies in advanced math, vol. 105, Cambridge University Press, Cambridge, 2006.
- B.Green, Additive Combinatorics, *lecture note* available online, https://people.maths.ox.ac.uk/greenbj/papers/additive-combinatorics.pdf

- M. Hochman, Self similar sets, entropy and additive combinatorics, *exposition article*, available online, https://arxiv.org/abs/1307.6399

4. **Mode of delivery:**

- Students need to read the materials and complete the problem assignments.
- Virtual Seminars via Zoom or Microsoft Teams.
- Face-to-face classes at Koc University.
- Virtual and face to face group meetings

5. **Period**: June 2025

   Virtual phase: 11.06-12.06 and 18.06.2025
   Physical phase: 22.06 - 29.06.2025

6. **Students:** Open to undergraduate students who have received at least 60 ECTS credits in mathematical courses or for graduate students.

7. **Workload**:

   - Independent work (Reading materials, homework problems) 20 h
   - Lectures 12 h (Virtual 4 h, Face to face 8 h)
   - Total number of hours: 32 h

8. **Assessment methods:**

   In order to pass this course, the following assessment criteria will be used:
   - Compulsory participation in the virtual and physical lectures
   - Compulsory reading and homework assignments

Grading: Pass/Fail or Satisfactory/Unsatisfactory

9. **Teaching language:** English

10. **ECTS:** 5 in total together with the other three topics:

- Theory of Stochastic Processes
- Diophantine m-tuples
- Fourier Analysis

in the Summer School. Total workload of the four topics is 4x32=128 h.

Topic IV: Diophantine m-tuples

Instructor: Assoc. Prof. Dr. Zrinka Franušić, Faculty of Science, University of Zagreb

1. **Course description:** Number theory is full of problems that seem easy but are very difficult to solve. Here we deal with the finite sets of numbers (mainly integers) with the property that the product of any two of its distinct elements increased by 1 is a perfect square, i.e. *Diophantine m-tuples*. These sets are named after Diophantus of Alexandria (3rd century) who found the first rational quadruple with this property. An interesting problem related to these sets is finding the upper bound for their size. In this regard, we will consider the extensibility of some Diophantine pairs and triples. Furthermore, some generalizations of the Diophantine m-tuples will be presented.
2. **Course goals:**

- To present modern methods used in solving the extension problem of Diophantine triples and pairs, such as methods for solving the system of simultaneous Pellian equations, Baker's theory of linear forms in logarithms, the hypergeometric method, and Baker-Davenport reduction.

3. **Timetable**

Date & Time	Content
Virtual phase	
11.06.2025 or 12.06.2025 (2 hrs) exact time to be announced later (TBA)	**Lecture 1: Review of Course Material** We introduce basic terms and notation, formulate problems and give a brief historical overview. We review basic concepts from elementary number theory necessary for the understanding of the lectured material.
18.06.2025 (2 hrs) exact time TBA	**Lecture 2: Introduction to the Topic** Approximation of irrational numbers by simple continued fractions. Periodic continued fractions. The connection between continued fractions and Pell's equations.
Physical phase at Koc University	
24.06.2025 (2 hrs) 15:00-17:00	**Lecture I:** The extensibility of the Diophantine pair. Solving Pellian (or generalized Pell) equation.
26.06.2025 (2 hrs) 11:00-13:00	**Lecture II:** A brief overview of methods and algorithms from Diophantine approximation. Linear form in logarithms of algebraic numbers.
27.06.2025 (2 hrs) 9:00-11:00	**Lecture III:** The extensibility of the Diophantine pair. Solving the system of Pellian equations. Baker-Davenport reduction.
28.06.2025 (2 hrs) 11:00-13:00	**Lecture IV:** The connection between $D(n)$-quadruples and the representation of n by the binary quadratic form $x^2 - y^2$. Polynomial formulas for $D(n)$-quadruples.

Recommended reading/textbook list for the course:

- A. Dujella, Number Theory, Školska knjiga, 2021.
- T. Nagell, Introduction to Number Theory, AMS, 1964.

4. **Mode of delivery:**

- Students need to read the materials and complete the problem assignments.
- Virtual Seminars via Zoom or Microsoft Teams.
- Face-to-face classes at Koc University.
- Virtual and face to face group meetings

5. **Period**: June 2025

   Virtual phase: 11.06-12.06 and 18.06.2025
   Physical phase: 22.06 - 29.06.2025

6. **Students:** Open to undergraduate students who have received at least 60 ECTS credits in mathematical courses or for graduate students.
7. **Workload**:

   - Independent work (Reading materials, homework problems) 20 h
   - Lectures 12 h (Virtual 4 h, Face to face 8 h)
   - Total number of hours: 32 h

8. **Assessment methods:**

In order to pass this course, the following assessment criteria will be used:

- Compulsory participation in the virtual and physical lectures
- Compulsory reading and homework assignments

Grading: Pass/Fail or Satisfactory/Unsatisfactory

9. **Teaching language:** English
10. **ECTS:** 5 in total together with the other three topics:

- Additive combinatorics
- Theory of stochastic processes
- Fourier analysis

in the Summer School. Total workload of the four topics is 4x32=128 h.

Programme Overview for 22-29 June 2025

Sunday June 22: Arrival to Koc University Campus

Time Slot	Monday June 23	Tuesday June 24	Wednesday June 25	Thursday June 26	Friday June 27	Saturday June 28	Sunday June 29
9:00-11:00	Topic I	Topic II	Topic I	Topic III	Topic IV	Topic III	Discussion
11:00-13:00	Topic II	Topic I	Topic II	Topic IV	Topic III	Topic IV	Discussion
13:00-15:00	LUNCH and Discussion	LUNCH and Discussion	LUNCH and Discussion	LUNCH and Discussion	LUNCH and Discussion	LUNCH and Discussion	LUNCH
15:00-17:00	Topic III	Topic IV	Free Afternoon	Topic I	Topic II	Question and Answer	Free Afternoon